Dimension of the product and classical formulae of dimension theory

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Abstract

Let $f : X \longrightarrow Y$ be a map of compact metric spaces. A classical theorem of Hurewicz asserts that $\dim X \leq \dim Y + \dim f$ where $\dim f = \sup\{\dim f^{-1}(y) : y \in Y\}$. The first author conjectured that $\dim Y + \dim f$ *in Hurewicz's theorem can be replaced by* $\sup\{\dim(Y \times f^{-1}(y)) : y \in Y\}$. We disprove this conjecture. As a by-product of the machinery presented in the paper we answer in negative the following problem posed by the first author: *Can for compact* X *the Menger-Urysohn formula* $\dim X \leq \dim A + \dim B + 1$ *be improved to* $\dim X \leq \dim(A \times B) + 1$ *?*

On a positive side we show that both conjectures holds true for compacta X satisfying the equality $dim(X \times X) = 2 \dim X$.

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1 Introduction

Throughout this paper we assume that maps are continuous and spaces are separable metrizable. We recall that a *compactum* means a compact metric space. By dimension of a space $\dim X$ we assume the covering dimension.

Clearly, the dimension of the product of two polyhedra equals the sum of the dimension: $\dim(K \times L) = \dim K + \dim L$. In 1930 Pontryagin discovered that this logarithmic law does not hold for compacta [16]. He constructed his famous Pontryagin surfaces Π_p indexed by prime numbers, $\dim \Pi_p = 2$, such that $\dim(\Pi_p \times \Pi_q) = 3$ whenever $p \neq q$. In the 80s the first author showed that the dimension of the product can deviate arbitrarily from the sum of the dimension. Namely, for any $n, m, k \in \mathbb{N}$ with

$$\max\{n, m\} + 1 \leq k \leq n + m$$

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there are compacta X_n and X_m of dimensions n and m respectively with $\dim(X_n \times X_m) = k$ [2]. We note that the inequality $\dim(X \times Y) \leq \dim X + \dim Y$ always holds true.

The first author conjectured that many classical formulas (inequalities) of dimension theory can be strengthen by replacing the sum of the dimensions by the dimension of the product. His believe was based on his results on the general position properties of compacta in euclidean spaces [5],[8]. Clearly, for two polyhedra K and L with transversal intersection in $\mathbb{R}^n$ we have $\dim(K \cap L) = n - (\dim K + \dim L)$. For compacta the corresponding formula is $\dim(X \cap Y) = n - \dim(X \times Y)$. In particular, two compacta X and Y in general position in $\mathbb{R}^n$ have empty intersection if and only if $\dim(X \times Y) < n$.

The next candidate for the improvement was the following classical theorem of Hurewicz.

Theorem 1.1 (**Hurewicz Theorem**) *Let $f : X \longrightarrow Y$ be a map of compacta. Then*

$$\dim X \leq \dim Y + \dim f$$

where $\dim f = \sup\{\dim f^{-1}(y) \mid y \in Y\}$.

We note that the Hurewicz theorem applied to the projection $X \times Y \to Y$ implies the inequality $\dim(X \times Y) \leq \dim X + \dim Y$. The first author proposed the following conjecture.

Conjecture 1.2 ([8]) *For a map of compacta $f : X \longrightarrow Y$*

$$\dim X \leq \sup\{\dim(Y \times f^{-1}(y)) \mid y \in Y\}.$$

Note that the Conjecture 1.2 holds true for nice maps like locally trivial bundles. It was known that the conjecture holds true when X is standard (compactum of type I in the sense of [12]). We call a compactum X *standard* if it has the property $\dim(X \times X) = 2\dim X$. It's not easy to come with an example of a compactum without this property. The Pontryagin surfaces satisfy it. First example of a non-standard compactum was constructed by Boltyanskii [1]. In this paper all non-standard compacta (compacta of type II in [12]) will be called *Boltyanskii compacta*. It is known that for all Boltyanskii compacta $\dim(X \times X) = 2\dim X - 1$.

In this paper we disprove Conjecture 1.2. We will refer to the maps providing counterexamples to the conjecture as *exotic maps*.

Positive results towards Conjecture 1.2 can be summarized in the following:

Theorem 1.3 *If a compactum X admits an exotic map $f : X \to Y$ then X is a Boltyanskii compactum. For every exotic map $f : X \to Y$ we have*

$$\dim X = \sup\{\dim(Y \times f^{-1}(y)) \mid y \in Y\} + 1.$$

Another classical result in Dimension Theory where the first author hoped to replace the sum of the dimensions by the dimension of the product was the Menger-Urysohn Formula.

Theorem 1.4 (**Menger-Urysohn Formula)** *Let $X = A \cup B$ be a decomposition of a space X. Then* $\dim X \leq \dim A + \dim B + 1$.

Problem 1.5 (**[5])** *Does the inequality* $\dim X \leq \dim(A \times B) + 1$ *hold true for an arbitrary decomposition of compact metric space $X = A \cup B$?*

In this paper we answer Problem 1.5 in the negative and, similarly to the terminology used above, we refer to the decompositions providing counterexamples to Problem 1.5 as *exotic decompositions*. To a certain extent exotic decompositions is a starting point of our construction of exotic maps.

Note that in the case of non-compact X a counter example to Problem 1.5 was constructed by Jan van Mill and Roman Pol. They proved the following.

Theorem 1.6 ([14]) *There is a 3-dimensional subset $X \subset \mathbb{R}^4$ admitting a decomposition $X = A \cup B$ such that* $\dim(A \times B)^n = 1$ *for every integer $n > 0$.*

Similarly to the case of Conjecture 1.2 the following facts were known about Problem 1.5.

Theorem 1.7 ([5])) *If a compactum X admits an exotic decomposition then X is a Boltyanskii compactum. For any exotic decomposition $X = A \cup B$ of a compactum X we have* $\dim X = \dim(A \times B) + 2$.

The main results of this paper are the following theorems.

Theorem 1.8 *Every finite dimensional Boltyanskii compactum X with* $\dim X \geq 5$ *admits an exotic decomposition.*

Theorem 1.9 *For every $n \geq 4$ there is an n-dimensional Boltyanskii compactum X admitting an exotic map $f : X \longrightarrow Y$ to a 2-dimensional compactum Y.*

Theorem 1.9 is derived from a more general result.

Theorem 1.10 *Every n-dimensional Boltyanskii compactum X with $n \geq 5$ and* $\dim_{\mathbb{Q}} X < n - 3$ *admits an exotic map $f : X \longrightarrow Y$ to an m-dimensional compactum Y with* $m = \dim_{\mathbb{Q}} X + 1$.

Note that no compactum of dim < 4 admits an exotic map and no compactum of dim $<$ 5 admits an exotic decomposition, see Section 4. A further development of the approach presented in the paper allows one to partially generalize Theorem 1.10 by showing that any finite dimensional Boltyanskii compactum X with $\dim X \geq 6$ admits an exotic map. This result is technically more complicated and will appear elsewhere. It still remains open whether any Boltyanski compactum of dimensions 4 and 5 admits an exotic map.

The paper is built as follows: Bockstein Theory is reviewed in Section 2; Section 3 is devoted to basic facts of Extension Theory with applications to Dimension Types; in Section 4 we consider the so-called compactly represented spaces, prove Theorem 1.8 and present short proofs for Theorems 1.3 and 1.7; and, finally, Theorems 1.9 and 1.10 are proved in Section 5.

2 Bockstein Theory

We recall some basic facts of Bockstein Theory. The first detailed presentation of the theory was given in the survey [12]. Since then it was evolved in many papers and surveys [2],[7],[6],[5],[17],[10]. Our presentation here has features of both point of view on the subject, classical and modern.

We remind that cohomology always means the Cech cohomology. Let G be an abelian group. The **cohomological dimension** $\dim_G X$ of a space X with respect to the coefficient group G does not exceed n, $\dim_G X \leq n$ if $H^{n+1}(X, A; G) = 0$ for every closed $A \subset X$. We note that this condition implies that $H^{n+k}(X, A; G) = 0$ for all $k \geq 1$ [12],[6]. Thus, $\dim_G X =$ the smallest integer $n \geq 0$ satisfying $\dim_G X \leq n$ and $\dim_G X = \infty$ if such an integer does not exist. Clearly, $\dim_G X \leq \dim_{\mathbb{Z}} X \leq \dim X$. Note that $\dim_G X = 0$ for a non-degenerate group G if and only if $\dim X = 0$.

Theorem 2.1 (**Alexandroff**) $\dim X = \dim_{\mathbb{Z}} X$ *if X is a finite dimensional space.*

Let $\mathcal{P}$ denote the set of all primes. The *Bockstein basis* is the collection of groups $\sigma = \{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Z}_{p^\infty}, \mathbb{Z}_{(p)} \mid p \in \mathcal{P}\}$ where $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ is the p-cyclic group, $\mathbb{Z}_{p^\infty} = \mathrm{dirlim}\mathbb{Z}_{p^k}$ is the p-adic circle, and $\mathbb{Z}_{(p)} = \{m/n \mid n$ is not divisible by $p\} \subset \mathbb{Q}$ is the p-localization of integers.

The Bockstein basis of an abelian group G is the collection $\sigma(G) \subset \sigma$ determined by the rule:

$\mathbb{Z}_{(p)} \in \sigma(G)$ if $G/\mathrm{Tor}G$ is not divisible by p;
$\mathbb{Z}_p \in \sigma(G)$ if p-$\mathrm{Tor}G$ is not divisible by p;
$\mathbb{Z}_{p^\infty} \in \sigma(G)$ if p-$\mathrm{Tor}G \neq 0$ is divisible by p;
$\mathbb{Q} \in \sigma(G)$ if $G/\mathrm{Tor}G \neq 0$ is divisible by all p.

Thus $\sigma(\mathbb{Z}) = \{\mathbb{Z}_{(p)} \mid p \in \mathcal{P}\}$.

Theorem 2.2 (**Bockstein Theorem**) *For a compactum X,*

$$\dim_G X = \sup\{\dim_H X : H \in \sigma(G)\}.$$

The Alexandroff and Bockstein theorems imply that for finite dimensional compacta X

$$\dim X = \max\{\dim_{\mathbb{Z}_{(p)}} X \mid p \in \mathcal{P}\}.$$

We call a space X *p-regular* if

$$\dim_{\mathbb{Z}_{(p)}} X = \dim_{\mathbb{Z}_p} X = \dim_{\mathbb{Z}_{p^\infty}} X = \dim_{\mathbb{Q}} X$$

and call it *p-singular* otherwise.

The restrictions on the values of cohomological dimension of a given space with respect to Bockstein groups usually are stated in the form of Bockstein inequalities [12]. Here we state them in a form of the equality and the alternative (see [6]).

Theorem 2.3 *I. For every p-singular space X and every prime p*

$$\dim_{\mathbb{Z}_{(p)}} X = \max\{\dim_{\mathbb{Q}} X, \dim_{\mathbb{Z}_{p^\infty}} X + 1\}.$$

II. (Alternative) For every p-singular space X and every prime p either

$$\dim_{\mathbb{Z}_{p^\infty}} X = \dim_{\mathbb{Z}_p} X \quad \textit{or} \quad \dim_{\mathbb{Z}_{p^\infty}} X = \dim_{\mathbb{Z}_p} X - 1.$$

In the first case of the alternative we call X p^+*-singular* and in the second, p^-*-singular*. Thus, the values of $\dim_F X$ for Bockstein fields $F \in \{\mathbb{Z}_p, \mathbb{Q}\}$ together with p-singularity types of X determine the value $\dim_G X$ for all groups.

We notice that he Alexandroff theorem, the Bockstein theorem, and Theorem 2.3 imply the following.

Corollary 2.4 *For every finite dimensional compactum X there is a field F such that* $\dim X \leq \dim_F X + 1$.

A function $f : \sigma \longrightarrow \mathbb{N} \cup \{0, \infty\}$ is called a *p-regular* if $f(\mathbb{Z}_{(p)}) = f(\mathbb{Z}_p) = f(\mathbb{Z}_{p^\infty}) = f(\mathbb{Q})$ and it is called *p-singular* if $f(\mathbb{Z}_{(p)}) = \max\{f(\mathbb{Q}), f(\mathbb{Z}_{p^\infty}) + 1\}$. A p-singular function f is called p^+-singular if $f(\mathbb{Z}_{p^\infty}) = f(\mathbb{Z}_p)$ and it is called p^-*-singular* if $f(\mathbb{Z}_{p^\infty}) = f(\mathbb{Z}_p) - 1$. A function $D : \sigma \longrightarrow \mathbb{N} \cup \{0, \infty\}$ is called a *dimension type* if for every prime p it is either p-regular or $p^\pm$-singular. For every space X the function $d_X : \sigma \longrightarrow \mathbb{N} \cup \{0, \infty\}$ defined as $d_X(G) = \dim_G X$ is a dimension type. If X is compactum d_X is called the *dimension type of* X. We denote $\dim D = \sup\{D(G) \mid G \in \sigma\}$.

Theorem 2.5 (**Dranishnikov Realization Theorem [2],[4]**) *For every dimension type D there is a compactum X with $d_X = D$ and* $\dim X = \dim D$.

Let D be a dimension type. We will use abbreviation $D(0) = D(\mathbb{Q})$, $D(p) = D(\mathbb{Z}_p)$. Additionally, if $D(p) = n \in \mathbb{N}$ we will write $D(p) = n^+$ if D is p^+-regular and $D(p) = n^-$ if it is p^--regular. For p-regular D we leave it without decoration: $D(p) = n$. Thus, any sequence of decorated numbers $D(p) \in \mathbb{N}$, where $p \in \mathcal{P} \cup \{0\}$ define a unique dimension type. There is a natural order on decorated numbers

$$\ldots < n^- < n < n^+ < (n+1)^- < \ldots \ .$$

Note that the inequality of dimension types $D \leq D'$ as functions on σ is equivalent to the family of inequalities $D(p) \leq D'(p)$ for the above order for all $p \in \mathcal{P} \cup \{0\}$. The natural involution on decorated numbers that exchange the decorations '+' and '-' keeping the base fixed defines an involution $*$ on the set of dimension types . Thus, $*$ takes p^+-singular function D to p^--singular D^* and vise versa.

By Alexandroff and Bockstein theorems it follows that for any compactum X of $\dim X = n < \infty$ there is a prime p such that $\dim X = \dim_{\mathbb{Z}_{(p)}} X$. Then either $d_X(0) = n$ or $d_X(p)$ equals one of the following: n or n^- or $(n-1)^+$. Let $\mathcal{P}_X$ denote the set of all such primes.

The Bockstein Product Theorem [12] gives the formulas for cohomological dimension of the product with respect to each of the groups $G \in \sigma$ which are huge for some of G. Here we state it in an alternative way (see [6],[17],[10]).

Theorem 2.6 (**Bockstein Product Theorem**) *For every field F and any two compacta,*

$$\dim_F(X \times Y) = \dim_F X + \dim_F Y.$$

For every prime p the type of p-singularity is preserved by multiplication by a p-regular compactum, and the following rule is applied in the remaining cases:
p^+-singular $\times$ p^+-singular $=$ p^+-singular;
p^--singular $\times$ $p^\pm$-singular$=$ p^--singular.

The product formula implies that an n-dimensional compactum X is a Boltyanskii compactum if and only if $d_X(0) < n$ and $d_X(p) = (n-1)^+$ for all $p \in \mathcal{P}_X$. For every $n \geq 2$ we denote by B_n, the "maximal" dimension type of Boltyanskii compacta of dimension n. Thus, $B_n(p) = (n-1)^+$ for all $p \in \mathcal{P}$ and $B_n(\mathbb{Q}) = n-1$. This implies that $B_n(\mathbb{Z}_{(p)}) = n$ for every prime p and $B_n(G) = n-1$ for all other groups in σ.

Corollary 2.7 ([**5**]) *For an n-dimensional compactum X the following are equivalent:*

- *X is a Boltyanskii compactum;*
- $\dim X > \dim_F X$ *for every field F;*
- *$d_X(G) \leq B_n(G)$ for all $G \in \sigma$.*

A finite dimensional compactum X is standard if and only if there is a field $F \in \sigma$ such that $\dim X = \dim_F X$*.*

Let D_1 and D_2 be dimension types. The dimension type $D_1 \boxplus D_1$ is defined by the formulas of the Bockstein Product Theorem: $(D_1 \boxplus D_2)(G) = \dim_G(X \times Y)$ with $\dim_G X$ and $\dim_G Y$ being replaced by $D_1(G)$ and $D_2(G)$ respectively for $G \in \sigma$ (see [7]). Thus we have that $d_{X\times Y} = d_X \boxplus d_Y$ for compacta X and Y. If $D_1(p) = n^{\epsilon_1}$ and $D_2(p) = m^{\epsilon_2}$ where ϵ_i is a decoration, i.e., '+' or '-' or empty, then

$$(D_1 \boxplus D_2)(p) = (n+m)^{\epsilon_1 \otimes \epsilon_2}$$

with the product of the signs $\epsilon_1 \otimes \epsilon_2$ defined by the Bockstein Product Theorem rule:

$$\epsilon \otimes \textit{empty} = \epsilon, \quad \epsilon \otimes \epsilon = \epsilon, \;\; \epsilon = \pm, \quad \textit{and} \quad + \otimes - = -.$$

By $D_1 + D_2$ and $D_1 \leq D_2$ we mean the ordinary sum and order relation when D_1 and D_2 are considered as just functions. Note that $D_1 + D_2$ is not always a dimension type but it is a dimension type, provided one of the summands is p-regular for all p. By 0 and 1 we denote the dimension types which send every $G \in \sigma$ to 0 and 1 respectively. Recall that $d_X = 0$ if and only if $\dim X = 0$ and

$$d_{X\times[0,1]} = d_X + 1.$$

The following inequality is an easy observation.

Proposition 2.8 *For any dimension types D_1 and D_2,*

$$D_1 \boxplus D_2 \leq (D_1^* \boxplus D_2^*)^*.$$

Proof. Clearly, we have the equality $(D_1 \boxplus D_2)(F) = (D_1^* \boxplus D_2^*)^*(F)$ for the fields. Thus, it suffices the check the inequality for the decorations. If $D_1^* \boxplus D_2^*$ is p^--singular, then the right hand part will have the decoration '+' and the inequality holds. If $D_1 \boxplus D_2$ is p^- singular, clearly the inequality holds. In the remaining case both D_1 and D_2 are p-regular and therefore, we have the equality. ■

3 Extension Theory

Cohomological Dimension is characterized by the following basic property: $\dim_G X \leq n$ if and only for every closed $A \subset X$ and a map $f : A \longrightarrow K(G, n)$, f continuously extends over X where $K(G, n)$ is the Eilenberg-MacLane complex of type (G, n) (we assume that $K(G, 0) = G$ with discrete topology and $K(G, \infty)$ is a singleton). This extension characterization of Cohomological Dimension gives a rise to Extension Theory (more general than Cohomological Dimension Theory) and the notion of Extension Dimension. The *extension dimension* of a space X is said to be dominated by a CW-complex K, written e-dim$X \leq K$, if every map $f : A \longrightarrow K$ from a closed subset A of X continuously extends over X. Thus $\dim_G X \leq n$ is equivalent to e-dim$X \leq K(G, n)$ and $\dim X \leq n$ is equivalent to e-dim$X \leq S^n$. For a dimension type D we denote $K(D) = \vee_{G\in\sigma} K(G, D(G))$. Then $d_X \leq D$ if and only e-dim$X \leq K(D)$.

Extension Dimension has many properties similar to Covering Dimension. For example: if e-dim$X \leq K$ then e-dim$A \leq K$ for every $A \subset X$ and if $X = \cup F_i$ is a countable union of closed subsets of X such that e-dim$F_i \leq K$ for every i then e-dim$X \leq K$. Let us list a few more basic results of Extension Theory.

Theorem 3.1 (**Olszewski Completion Theorem [15]**) *Let K be a countable CW-complex and* e-dim$X \leq K$*. Then there is a completion of X dominated by K.*

Corollary 3.2 *For every separable metric space X there is a completion X' such that for all $G \in \sigma$,*

$$\dim_G X' = \dim_G X.$$

We note that for finite dimensional X this corollary follows from the theory of test spaces [11], [12] and the well-known fact that for every compactum C there is a completion X' of X with $\dim(X' \times C) = \dim(X \times C)$ (see for example Proposition 6.2 in[5]).

Theorem 3.3 (**Dranishnikov Extension Theorem [3],[9]**) *Let K be a CW-complex and X a space. Then*

(i) $\dim_{H_n(K)} X \leq n$ *for every $n \geq 0$ if* e-dim$X \leq K$*;*

(ii) e-dim$X \leq K$ *if K is simply connected, X is finite dimensional and* $\dim_{H_n(K)} X \leq n$ *for every $n \geq 0$*

We remind that $H_*(K)$ denotes the reduced homology.

Let K be a CW-complex. For $G \in \sigma$ denote $n_G(K) = \min\{n : G \in \sigma(H_n(K))\}$ or $n_G(K) = \infty$ if the set defining $n_G(K)$ is empty. If X is a compactum and e-dim$X \leq K$ then, by the Dranishnikov Extension Theorem and the Bockstein Theorem, we have that $\dim_G X \leq n_G(K)$ for every $G \in \sigma$.

Theorem 3.4 (**Dydak Union Theorem [9]**) *Let K and L be CW-complex and $X = A \cup B$ a decomposition of a space X such that* e-dim$A \leq K$ *and* e-dim$B \leq L$. *Then* e-dim$X \leq K * L$.

We recall that $K * L = \Sigma(K \wedge L)$.

Theorem 3.5 (**Dranishnikov Decomposition Theorem [4]**) *Let K and L be countable CW-complexes and X a compactum such that* e-dim$X \leq K * L$. *Then there is a decomposition $X = A \cup B$ of X such that* e-dim$A \leq K$ *and* e-dim$B \leq L$.

Let D_1 and D_2 be dimension types such that at least one of them is different from 0 and $X = A \cup B$ a decomposition of a compactum X such that $d_A \leq D_1$ and $d_B \leq D_1$. By the Dydak Union Theorem, $d_X \leq K(D_1) * K(D_2)$. Then $\dim_G X \leq n_G(K(D_1) * K(D_2)) = n_G(\Sigma(K(D_1) \wedge K(D_2))) = n_G(K(D_1) \wedge K(D_2)) + 1$, $G \in \sigma$. Thus one can estimate the dimension type of X by computing the numbers $n_G(K(D_1) \wedge K(D_2))$, $G \in \sigma$. This computation was done by Dranishnikov [5]. We denote by $D_1 \oplus D_2$ the biggest dimension type such that $(D_1 \oplus D_2)(G) \leq n_G(K(D_1) \wedge K(D_1)), G \in \sigma$ and set $D_1 \oplus D_2 = 0$ for $D_1 = D_2 = 0$. The following can be easily derived from Dranishnikov's computation [5]:

Theorem 3.6 *Let D_1 and D_2 be dimension types. Then*

$$D_1 \oplus D_2 = (D_1^* \boxplus D_2^*)^*.$$

Thus if $X = A \cup B$ is a decomposition of a compactum X with $d_A \leq D_1$ and $d_B \leq D_2$ then $d_X \leq D_1 \oplus D_2 + 1$.

Now assume that X is a finite dimensional compactum and D_1 and D_2 are dimension types such that $d_X \leq D_1 \oplus D_2 + 1$. If $D_1 = D_2 = 0$ then $\dim X \leq 1$ and for a decomposition $X = A \cup B$ into 0-dimensional subsets we obviously have $d_A \leq D_1$ and $d_B \leq D_2$. If at least one of D_1 and D_2 is different from 0 then $K(D_1) * K(D_2)$ is simply connected. Then, by the Bockstein Theorem and the Dranishnikov Extension Theorem, e-dim$X \leq K(D_1) * K(D_2)$ and, by the Dranishnikov Decomposition Theorem, there is a decomposition $X = A \cup B$ of X with e-dim$A \leq K(D_1)$ and e-dim$B \leq K(D_2)$ and, hence, $d_A \leq D_1$ and $d_B \leq D_2$. Thus we can summarize

Corollary 3.7 *Let X be a compactum and D_1 and D_2 dimension types:*

(i) if $X = A \cup B$ is a decomposition with $d_A \leq D_1$ and $d_B \leq D_2$ then $d_X \leq D_1 \oplus D_2 + 1$;

(ii) if X is finite dimensional and $d_X \leq D_1 \oplus D_2 + 1$ then there is a decomposition $X = A \cup B$ such that $d_A \leq D_1$ and $d_B \leq D_2$.

Note that

$$(D_1 \boxplus D_2)(F) = (D_1 \oplus D_2)(F) = D_1(F) + D_2(F)$$

for any dimension types D_1, D_2 and any field $F \in \sigma$.

Proposition 3.8 *Let the dimension types D_1, D_2, D_1' and D_2' satisfy $D_1 \leq D_1'$ and $D_2 \leq D_2'$. Then $D_1 \boxplus D_2 \leq D_1' \boxplus D_2'$ and $D_1 \oplus D_2 \leq D_1' \oplus D_2'$*

Proof. The first inequality is standard and it easy follows from the definitions. The second inequality follows from Theorem 3.6. ■

It turns out that the operation $\oplus$ nicely fits in the translation of some mapping theorems by Levin and Lewis [13] to the language of dimension types.

Theorem 3.9 (**Levin-Lewis [13]**) *Let $f : X \longrightarrow Y$ be a map of compacta and let K and L be CW-complexes such that* e-dim$f \leq K$ *and* e-dim$Y \leq L$. *Then $X \times [0,1]$ decomposes into $X \times [0,1] = A \cup B$ such that* e-dim$A \leq K$ *and* e-dim$B \leq L$.

Theorem 3.10 (**Levin-Lewis [13]**) *Let $f : X \longrightarrow Y$ be a map of compacta and, K a countable CW-complexes such that* e-dim$f \leq \Sigma K$ *and Y is finite dimensional. Then*

(i) there is a σ-compact set $A \subset X$ such that e-dim$A \leq K$ *and* $\dim f|_{X \setminus A} \leq 0$;

(ii) there is a map $g : X \longrightarrow [0,1]$ such that for the map $(f,g) : X \longrightarrow Y \times [0,1]$ we have e-dim$(f,g) \leq K$.

Let $f : X \longrightarrow Y$ be a map. For a group G we denote $\dim_G f = \sup\{\dim_G f^{-1}(y) : y \in Y\}$ and for a CW-complex K we say that e-dim$f \leq K$ if e-dim$f^{-1}(y) \leq K$ for every $y \in Y$. Similarly, for a dimension type D we say that $d_f \leq D$ if $d_{f^{-1}(y)} \leq D$ for every $y \in Y$. Theorems 3.9 and 3.10 can be translated to dimension types as follows.

Corollary 3.11 *Let $f : X \longrightarrow Y$ be a map of compacta and let D_1 and D_2 be dimension types such that $d_f \leq D_1$ and $d_Y \leq D_2$. Then $d_X \leq D_1 \oplus D_2$. Moreover, if $F \in \sigma$ is a field then $\dim_F X \leq \dim_F f + \dim_F Y$.*

Proof. Apply Theorem 3.9 for $K = K(D_1)$ and $L = K(D_2)$ to get a decomposition $X \times [0,1] = A \cup B$ with e-dim$A \leq K(D_1)$ and e-dim$B \leq K(D_2)$. Then $d_A \leq D_1$, $d_B \leq D_2$ and, by 3.7, $d_X + 1 = d_{X \times [0,1]} \leq D_1 \oplus D_2 + 1$ and hence $d_X \leq D_1 \oplus D_2$.

Now let $F \in \sigma$ be a field, $n = \dim_F f$, $m = \dim_F Y$ and let $K = K(F,n)$ and $L = K(F,m)$. Then, by the reasoning we just used, there is a decomposition $X \times [0,1] = A \cup B$ such that e-dim$A \leq K$ and e-dim$B \leq L$ and, by Corollary 3.7, $d_X + 1 \leq d_A \oplus d_B + 1$. Hence $\dim_F X = d_X(F) \leq (d_A \oplus d_B)(F) = d_A(F) + d_B(F) = \dim_F A + \dim_F B \leq n + m = \dim_F f + \dim_F Y$. ■

Corollary 3.12 *Let $f : X \longrightarrow Y$ be a map of finite dimensional compacta and D a dimension type such that $d_f \leq D + 1$. Then*

(i) there is a σ-compact set $A \subset X$ such that $d_A \leq D$ and $\dim(f|_{X \setminus A}) \leq 0$;

(ii) there is a map $g : X \longrightarrow [0,1]$ such that for the map $(f,g) : X \longrightarrow Y \times [0,1]$ we have $d_{(f,g)} \leq D$.

Proof. By Corollary 3.7 we have that each fiber $f^{-1}(y)$ decomposes into $f^{-1}(y) = \Omega_1 \cup \Omega_2$ with $d_{\Omega_1} \leq 0$ and $d_{\Omega_2} \leq D$. Then e-dim$\Omega_1 \leq S^0$, e-dim$\Omega_2 \leq K(D)$ and, by the Dydak Union Theorem, e-dim$f^{-1}(y) \leq S^0 * K(D) = \Sigma K(D)$.

Thus, e-dim$f \leq \Sigma K(D)$ and the corollary follows from Theorem 3.10. ■

We end this section with the following observation.

Proposition 3.13 *Let X be a finite dimensional compactum and $n > 0$. Then* $\dim_{\mathbb{Q}} X \leq n$ *if and only if for every closed subset A of X and every map $f : A \longrightarrow S^n$ there is a map $g : S^n \longrightarrow S^n$ of non-zero degree such that $g \circ f : X \longrightarrow S^n$ continuously extends over X.*

Proof. Let $M(\mathbb{Q}, n)$ be a Moore space of type $(\mathbb{Q}, n)$. Represent $M(\mathbb{Q}, n)$ as the telescope of a sequence of maps $\phi_i : S^n \longrightarrow S^n$ such that $\deg \phi_i = i, i > 0$. Note that $M(\mathbb{Q}, 1) = K(\mathbb{Q}, 1)$. By the Dranishnikov Extension Theorem e-dim$X \leq M(\mathbb{Q}, n)$ is equivalent to $\dim_{\mathbb{Q}} X \leq n$ for $n \geq 2$. Thus e-dim$X \leq M(\mathbb{Q}, n)$ is equivalent to $\dim_{\mathbb{Q}} X \leq n$ for every $n > 0$.

Assume that $\dim_{\mathbb{Q}} X \leq n$. Consider f as a map to the first sphere of $M(\mathbb{Q}, n)$ and continuously extend f to $f' : X \longrightarrow M(\mathbb{Q}, n)$. Then $f'(X)$ is contained in a finite subtelescope M' of $M(\mathbb{Q}, n)$. Let $r : M' \longrightarrow S^n$ be the natural retraction to the last sphere of M'. Then g can be taken as r restricted to the first sphere of M'.

Now we will show the other direction of the proposition. Take a map $\psi : A \longrightarrow M(\mathbb{Q}, n)$ from a closed subset A of X. Then $\psi(A)$ is contained in a finite subtelescope of $M(\mathbb{Q}, n)$. Assume that that this subtelescope ends at the i-th sphere of $M(\mathbb{Q}, n)$. Then ψ can be homotoped to a map $f : A \longrightarrow S^n$ to the i-th sphere of $M(\mathbb{Q}, n)$. Let $g : S^n \longrightarrow S^n$ be a map of degree $d > 0$ such that $g \circ f$ extends over X. Consider the subtelescope M' of $M(\mathbb{Q}, n)$ starting at the i-th sphere and ending at the $(i + d)$-sphere of $M(\mathbb{Q}, n)$ and let $r : M' \longrightarrow S^n$ be the natural retraction of M' to the last sphere of M'. Then the degree of r restricted to the i-the sphere of $M(\mathbb{Q}, n)$ is divisible by d and hence $r \circ f$ factors up to homotopy through $g \circ f$. Since $r \circ f$ and f are homotopic as maps to M' we get that f extends over X as a map to M' and therefore ψ extends as well. ■

4 Proofs of Theorems 1.3, 1.7 and 1.8

A space X is called *compactly represented* if for every $G \in \sigma \cup \{\mathbb{Z}\}$ there is a compactum $C \subset X$ such that $\dim_G C = \dim_G X$. We say that a space X is *compactly represented by a subset* $A \subset X$ if X is compactly represented and the compacta C witnessing that can be chosen to be subsets of A. Note that any σ-compact set is compactly represented. We say that a space X is *dimensionally dominated by a space* Y if $\dim_G X \leq \dim_G Y$ for every $G \in \sigma \cup \{\mathbb{Z}\}$. It follows from the Olszewski Completion Theorem that for a σ-compact subset A of a compactum X and a space Y there is a G_δ-subset $A' \subset X$ such that $A \subset A'$, A' is compactly represented by A and $A' \times Y$ is dimensionally dominated by $A \times Y$. Moreover, if Y is also σ-compact then we may assume that $A' \times Y$

is compactly represented by $A \times Y$. Note that $\dim_{\mathbb{Z}} X = \sup\{\dim_G X : G \in \sigma\}$ if X is compactly represented and $d_{X\times Y} = d_X \boxplus d_Y$ if X, Y and $X \times Y$ are compactly represented.

We say that a decomposition $X = A \cup B$ of a space X is a *compactly represented decomposition* if A, B and $A \times B$ are compactly represented and we say that a decomposition $X = A' \cup B'$ is *dimensionally dominated by a decomposition* $X = A \cup B$ if $\dim_G A \leq \dim_G A'$, $\dim_G B \leq \dim_G B'$ and $\dim_G(A \times B) \leq \dim_G(A' \times B')$ for every $G \in \sigma \cup \{\mathbb{Z}\}$.

The following proposition can be easily derived from the proof of Proposition 6.3 of [5].

Proposition 4.1 *Let X be a compactum, and $X = A \cup B$ a decomposition. Then there is a decomposition $X = A' \cup B'$ such that A' is σ-compact, $B' = X \setminus A'$ and the decomposition $X = A' \cup B'$ is dimensionally dominated by the decomposition $X = A \cup B$.*

We need a stronger version of Proposition 4.1.

Proposition 4.2 *Let X be a compactum. For any decomposition $X = A \cup B$ of X there is a compactly represented decomposition $X = A' \cup B'$ such that A' is σ-compact, $B' = X \setminus A'$ and the decomposition $X = A' \cup B'$ is dimensionally dominated by the decomposition $X = A \cup B$.*

Proof. By Proposition 4.1, we can assume that B is σ-compact and $A = X \setminus B$. Let B_1 be a G_δ-subset of X such that $B \subset B_1$, B_1 is compactly represented by B and $A \times B_1$ is dimensionally dominated by $A \times B$. Set $A_1 = X \setminus B_1$. Then there is a G_δ-subset B_2 of X such that $B \subset B_2 \subset B_1$, B_2 is compactly represented by B and $A_1 \times B_2$ is compactly represented by $A_1 \times B$. Proceed by induction and construct for every i a G_δ-set B_i and a σ-compact set $A_i = X \setminus B_i$ such that

(i) $B \subset B_{i+1} \subset B_i$;
(ii) B_i is compactly represented by B;
(iii) $A_i \times B_{i+1}$ is compactly represented by $A_i \times B$.

Then for $B' = \cap B_i$ and $A' = \cup A_i$ we have that $X = A' \cup B'$, $A' \subset A$ $B \subset B'$, A' is σ-compact, B' is G_δ, B' is compactly represented by B, $A' \times B'$ is compactly represented by $A' \times B$. Recall that $A_i \times B' \subset A \times B_1$ and $A \times B_1$ is dimensionally dominated by $A \times B$. Thus $A' \times B'$ is dimensionally dominated by $A \times B$ and the proposition follows. ∎

Proof of Theorem 1.3. Let $f : X \longrightarrow Y$ be an exotic map of compacta. Then for every field $F \in \sigma$ and every $y \in Y$ we have

$$\dim X > \dim(f^{-1}(y) \times Y) \geq \dim_F(f^{-1}(y) \times Y) = \dim_F f^{-1}(y) + \dim_F Y.$$

Hence

$$\dim X > \sup_{y\in Y}\{\dim(f^{-1}(y)\times Y)\} \geq \sup_{y\in Y}\{\dim_F(f^{-1}(y)\times Y)\} + \dim_F Y = \dim_F f + \dim Y.$$

Then, by Corollary 3.12, $\dim_F f + \dim Y \geq \dim_F X$ and hence

$$\dim X > \sup\{\dim(f^{-1}(y)\times Y) : y \in Y\} \geq \dim_F X$$

for every field $F \in \sigma$. Thus, by Corollary 2.7, we conclude that X is a Boltyanskii compactum and $\dim X = \sup\{\dim(f^{-1}(y)\times Y) \mid y \in Y\} + 1$. ■

Proof of Theorem 1.7. let $X = A \cup B$ be an exotic decomposition of a compactum X. By Proposition 4.2 we may assume that $X = A \cup B$ is a compactly represented decomposition. Then for every field $F \in \sigma$ we have

$$\dim_F X \leq \dim_F A + \dim_F B + 1 = \dim_F(A\times B) + 1 \leq \dim(A\times B) + 1 \leq \dim X - 1.$$

Thus $\dim X \geq \dim_F X + 1$ for every field $F \in \sigma$. Then, by Corollary 2.7, X is a Boltyanskii compactum and there is a field F such that $\dim_F X + 1 = \dim X$. Hence $\dim X = \dim(A\times B) + 2$ and the theorem follows. ■

Proof of Theorem 1.8. Let X be an n-dimensional Boltyanskii compactum with $n \geq 5$. Define the dimension types D_1 and D_2 by $D_1(p) = 2^-$, $D_1(\mathbb{Q}) = 1$ and $D_2(p) = (n-4)^+$, $D_2(\mathbb{Q}) = n-3$ for all primes p. Then

$$(D_1 \oplus D_2)(p) = (2^+ \boxplus (n-4)^-)^* = ((n-2)^-)^* = (n-2)^+$$

for all p and $(D_1\oplus D_2)(\mathbb{Q}) = D_1(\mathbb{Q}) + D_2(\mathbb{Q}) = n-2$. Thus, $D_1 \oplus D_2 = B_{n-1}$ where B_{n-1} is the maximal Boltyanskii dimension type of dimension $n-1$. Since $(D_1 \boxplus D_2)(p) = (n-2)^-$ and $(D_1 \boxplus D_2)(\mathbb{Q}) = n-2$, we obtain that $(D_1 \boxplus D_2)(\mathbb{Z}_{(p)}) = n-2$ and hence, $\dim(D_1 \boxplus D_2) \leq n-2$. By Corollary 2.7, $d_X \leq B_n = B_{n-1} + 1 = D_1 \oplus D_2 + 1$ and, by Corollary 3.7, there is a decomposition $X = A \cup B$ such that $d_A \leq D_1$ and $d_B \leq D_2$. By Proposition 4.2 we can assume that $X = A\cup B$ is a compactly represented decomposition. Then $d_{A\times B} \leq D_1 \boxplus D_2$ and

$$\dim(A\times B) = \dim_{\mathbb{Z}}(A\times B) = \dim d_{A\times B} \leq \dim(D_1 \boxplus D_2) \leq n-2.$$

Thus $X = A \cup B$ is an exotic decomposition and the theorem follows. ■

Note that for compacta X and Y with $\dim Y \geq 1$ we always have $\dim(X\times Y) \geq \dim X + 1$. This property immediately implies that no compactum of dimension≤ 3 admits an exotic map. Together with Proposition 4.2 this property also implies that no compactum of dimension≤ 4 admits an exotic decomposition.

5 Proofs of Theorems 1.9 and 1.10

For $n \geq 5$ and $n-3 \geq m \geq 2$ consider the dimension types D, D_1 and D_2 defined by $D(\mathbb{Q}) = D_1(\mathbb{Q}) = m-1, D_2(\mathbb{Q}) = n-m-1$ and for every $p \in \mathcal{P}$,

$$D(p) = (n-1)^+, \qquad D_1(p) = m^-, \qquad D_2(p) = (n-m-2)^+.$$

Note that $(D_1 \oplus D_2)(p) = (n-2)^+$ and $(D_1 \boxplus D_2)(p) = (n-2)^-$. Hence, $D \leq D_1 \oplus D_2 + 1$ and $\dim(D_1 \boxplus (D_2+1)) = n-1$.

Note that for an n-dimensional Boltyanskii compactum with $n \geq 5$ and $m = \dim_{\mathbb{Q}} X + 1 \leq n-3$ we have $d_X \leq D$. Then Theorem 1.10 immediately follows from the following proposition.

Proposition 5.1 *Every n-dimensional compactum X with $d_X \leq D$ admits a map $f : X \longrightarrow Y$ to an m-dimensional compactum Y such that $d_Y \leq D_1$ and $d_f \leq D_2 + 1$. Thus for every $y \in Y$,*

$$\dim(f^{-1}(y) \times Y) \leq \dim(D_1 \boxplus (D_2+1)) = n-1$$

and hence f is an exotic map.

All the cases of Theorem 1.9, except $n = 4$, are covered by Theorem 1.10 for $m = 2$. Let us show that the missing case $n = 4$ also follows from Proposition 5.1.

Proof of Theorem 1.9 (the missing case). Consider the map $f : X \longrightarrow Y$ constructed in Proposition 5.1 for $n = 5$ and $m = 2$. By Theorem 3.12, there is a map $g : X \longrightarrow [0,1]$ such that the map $(f,g) : X \longrightarrow Y \times [0,1]$ is of dimension type $d_{(f,g)} \leq D_2$. By the Hurewicz Theorem there is $t \in [0,1]$ such that $X' = g^{-1}(t)$ is of dim ≥ 4. Let $f' : X' \longrightarrow Y$ be the map $(f,g)|_{X'}$ followed by the projection from $Y \times [0,1]$ to $[0,1]$ and let $Y' = f'(X')$. Then $d_{f'} \leq D_2$ and $d_{Y'} \leq D_1$. Thus $\dim f' \leq 2$ and $\dim Y' \leq 2$ and since $\dim X' \geq 4$ we get, by the Hurewicz Theorem, that $\dim X' = 4$ and $\dim f' = \dim Y' = 2$. Note that $\dim(D_1 \boxplus D_2) = 3$ and hence $f' : X' \longrightarrow Y'$ is an exotic map we are looking for. $\blacksquare$

In the proof of Proposition 5.1 we will use the following.

Proposition 5.2 *Let X be a compactum, M an m-dimensional manifold possibly with boundary, A a σ-compact subset of X with $\dim A \leq m$, F a closed subset of X and $f : X \longrightarrow M$ a map which is 0-dimensional on $A \cap F$. Then f can be arbitrarily closely approximated by a map $f' : X \longrightarrow M$ such that f' is 0-dimensional on A and f' coincides with f on F.*

Proof. This a typical application of the Baire Category Theorem. The following fact is well known:

(1) For every m-dimensional compactum A the set G of 0-dimensional maps $f : A \to \mathbb{R}^m$ is dense G_δ in the space of all continuous maps $C(A, \mathbb{R}^m)$ given the uniform convergence topology.

For the proof we present G as the intersection of sets W_n of maps $f : A \to \mathbb{R}^m$ such that $diam\ C < 1/n$ for all components C of the preimage $f^{-1}(x)$ for all x. It is easy to see that each W_n is open. One way to show that W_n is dense is first to approximate a given map $f : A \to \mathbb{R}^m$ by a composition $f' \circ q$ where $q : A \to K^m$ is an $(1/n)$-map to an m-dimensional simplicial complex and then to approximate f' by a 0-dimensional map $g : K^m \to \mathbb{R}^m$. The latter can be obtained by a proper perturbation of all vertices of sufficiently small subdivision of K^m in $\mathbb{R}^m$ and by taking the corresponding piece-wise linear map g.

We note that $\mathbb{R}^m$ can be replaced by the half space $\mathbb{R}^m_+$ in this proof.

The above statement can be generalized to the following:

(2) For every compact metric pair (X, A) with m-dimensional A the set G of maps $f : (X, A) \to (M, B)$ with 0-dimensional restriction $f|_A$ is dense G_δ in the space of all continuous maps of pairs $C((X, A), (M, B))$ where M is compactum and $B \subset M$ is an open set homeomorphic to $\mathbb{R}^m$ or to $\mathbb{R}^m_+$.

Note that $C((X, A), (M, B))$ is open in $C(X, A)$ and hence is complete. Then the above argument works for this statement as well.

As a corollary we obtain the following:

(3) For every compact metric pair (X, A) with m-dimensional A and an m-dimensional manifold with boundary M every continuous map $f : X \to M$ can be approximated by maps 0-dimensional on A.

To derive it from the above we consider a finite cover $B_1, \dots B_k$ of $f(A)$ by open sets homeomorphic to $\mathbb{R}^m$ or $\mathbb{R}^m_+$ and a partition $A = A_1 \cup \dots \cup A_k$ into closed subsets such that $f(A_i) \subset B_i$. Let

$$W = W(\{A_i\}, \{B_i\}) = \{g : X \to M \mid g(A_i) \subset B_i, i = 1, \dots, k\}$$

be a corresponding neighborhood of f in the compact-open topology. Then the set $W \cap (\cap_i G_i)$ is dense G_δ in W where G_i the set of maps $g : (X, A_i) \to (M, B_i)$ which are 0-dimensional on A_i.

We note that the compactness of A in this statement can be replaced by σ-compactness.

Finally, to obtain the statement of the proposition we consider a compact subset $F \subset X$ with a fixed map $f_0 : F \to M$ which is 0-dimensional on $F \cap A$. Let $A' = A \setminus F$. Note that $A' = \cup A_i$ is the countable union of compact sets A_i. Now we prove the statement (3) for A' in the complete metric space $C(X, M; F, f_0) = \{f : X \to M \mid f|_F = f_0\}$ using the same proof. By the countable union theorem any map $f \in C(X, M; F, f_0)$ which is 0-dimensional on A' is also 0-dimensional on A. ■

Proof of Proposition 5.1. Since $D \le D_1 \oplus D_2 + 1$, by Corollary 3.7 there is a decomposition $X = A \cup B$ such that $d_A \le D_1$ and $d_B \le D_2$. By the corollary of Olszewski Completion Theorem we may assume that B is G_δ. Thus replacing A by $X \setminus B$ we assume that A is σ-compact. Represent $A = \cup A_i$ as a countable union compact subsets $A \subset X$

such that $A_i \subset A_{i+1}, i = 1, 2, \ldots$. Note that $\dim A \leq \dim D_1 = m$.

We will construct for each i an m-dimensional simplicial complex Y_i, a bonding map $\omega_i^{i+1} : Y_{i+1} \longrightarrow Y_i$ and a map $\phi_i : X \longrightarrow Y_i$. We fix metrics in X and in each Y_i and with respect to these metrics we determine $0 < \epsilon_i < 1/2^i$ such that the following properties will be satisfied:

(i) ϕ_i is 0-dimensional on A and for every open set $U \subset Y_i$ with $\mathrm{diam} U < 2\epsilon_i$ the set $\phi_i^{-1}(U) \cap A_i$ splits into disjoint sets open in A_i and of diam $\leq 1/i$;

(ii) $\mathrm{dist}(\omega_j^{i+1} \circ \phi_{i+1}, \omega_j^i \circ \phi_i) < \epsilon_j/2^i$ for $i \geq j$ where $\omega_i^j = \omega_{j-1}^j \circ \ldots \circ \omega_i^{i+1} : Y_j \longrightarrow Y_i$ for $j > i$ and $\omega_i^i = id : Y_i \longrightarrow Y_i$.

The construction will be carried out so that for $Y = \mathrm{invlim}(Y_i, \omega_i^{i+1})$ we have $\dim_{\mathbb{Q}} Y \leq m-1$. Let us first show that the proposition follows from this construction. Denote $f_i = \lim_{j\to\infty} \omega_i^j \circ \phi_j : X \longrightarrow Y_i$. From (ii) it follows that f_i is well-defined, continuous and $\mathrm{dist}(f_i, \phi_i) \leq \epsilon_i$. From the definition of f_i it follows that $f_j \circ f_i^j = f_i$. Hence the maps f_i define the corresponding map $f : X \longrightarrow Y$ such that $\omega_i \circ f = f_i$ where $\omega_i : Y \longrightarrow Y_i$ is the projection. Then it follows from (i) that for every $y \in Y_i$ the set $f_i^{-1}(y) \cap A_i$ splits into finitely many disjoint sets closed in A_i and of diam $\leq 1/i$. This implies that for every $y \in Y$ we have that $\dim(f^{-1}(y) \cap A_i) \leq 0$ and hence $\dim(f^{-1}(y) \cap A) \leq 0$. Then, by Corollary 3.7, $d_f \leq D_2 + 1$. Since $\dim Y_i \leq m$ we have $\dim Y \leq m$ and since $\dim D_2 + 1 = n - m$, Hurewicz Theorem implies that $\dim Y = m$. The condition $\dim_{\mathbb{Q}} Y \leq m-1$ and the formula for the cohomological dimension with respect to $\mathbb{Z}_{(p)}$ imply that $\dim_{\mathbb{Z}_{p^\infty}} Y \leq m-1$ for both p-regular and p-singular cases. Therefore, $d_Y \leq D_1$. Thus, for every $y \in Y$ we have

$$\dim(f^{-1}(y) \times Y) \leq \dim(D_1 \boxplus (D_2 + 1)) = n - 1$$

and hence f is an exotic map.

Now we return to our construction. Let Y_1 be an m-simplex and let $\phi_1 : X \longrightarrow Y$ be any map which is 0-dimensional on A. Then one can choose $0 < \epsilon_1 < 1/2$ so that (i) holds for $i = 1$. Assume that the construction is completed for i and proceed to $i+1$ as follows. Take a triangulation of Y_i so fine that for every simplex Δ of Y_i we have that $\mathrm{diam}(\omega_j^i(\Delta)) < \epsilon_j/2^i$ for every j such that $i \geq j \geq 1$. For each m-simplex Δ of Y_i consider a small ball D centered at the barycenter of Δ and not touching ∂D. Recall that $\dim_{\mathbb{Q}} X = m-1$. Then, by Proposition 3.13, for $\phi_i|_{\ldots} : \phi_i^{-1}(\partial D) \longrightarrow \partial D$ there is a map $\psi_{\partial D} : \partial D \longrightarrow \partial D$ of non-zero degree such that $\psi_{\partial D} \circ \phi_i|_{\ldots}$ extends over $\phi_i^{-1}(D)$ to a map $\phi_D : \phi_i^{-1}(D) \longrightarrow \partial D$. Clearly we can assume that $\psi_{\partial D}$ is 0-dimensional (even finite-to-one). Denote by $\tilde{Y}_{i+1}$ the quotient space of Y_i obtained by removing for each m-simplex Δ of Y_i the interior $\mathrm{Int}D$ of the ball D and identifying the points of ∂D according to the map $\psi_{\partial D}$. We consider $\partial\Delta$ also as a subset of $\tilde{Y}_{i+1}$ and we denote by Y_Δ the subspace of $\tilde{Y}_{i+1}$ obtained from Δ. Let $\tilde{\omega}_i^{i+1} : \tilde{Y}_{i+1} \longrightarrow Y_i$ be any map such that $\tilde{\omega}_i^{i+1}$ is 1-to-1 over the $(m-1)$-skeleton of Y_i and $\tilde{\omega}_i^{i+1}$ sends Y_Δ to Δ for every m-simplex Δ of Y_i. The

map ϕ_i and the maps ϕ_D naturally define the corresponding map $\tilde{\phi}_{i+1} : X \longrightarrow \tilde{Y}_{i+1}$ which coincides with ϕ_i on $\phi_i^{-1}(\partial\Delta)$ for every m-simplex Δ of Y_i. Note that $\tilde{\phi}_{i+1}$ is 0-dimensional on $A \cap \phi_i^{-1}(\Delta \setminus \mathrm{Int} D)$ for every m-simplex Δ in Y_i. Let a space $Y_{i+1} \supset \tilde{Y}_{i+1}$ be obtained from $\tilde{Y}_{i+1}$ by attaching for every sphere $\psi_{\partial D}(\partial D) \subset \tilde{Y}_{i+1}$ the manifold $\psi_{\partial D}(\partial D) \times [0,1]$ by identifying $\psi_{\partial D}(\partial D) \times 0$ with $\psi_{\partial D}(\partial D)$ and let $\pi_{i+1} : Y_{i+1} \longrightarrow \tilde{Y}_{i+1}$ be a retraction projecting each manifold $\psi_{\partial D}(\partial D) \times [0,1]$ to $\psi_{\partial D}(\partial D)$. Then, by Proposition 5.2, for every m-simplex Δ of Y_i we can extend $\tilde{\phi}_{i+1}$ restricted to $\phi_i^{-1}(\Delta \setminus \mathrm{Int} D)$ over $\phi_i^{-1}(\Delta)$ to a map being 0-dimensional on $A \cap \phi_i^{-1}(\Delta)$ and this way we define the map $\phi_{i+1} : X \longrightarrow Y_{i+1}$ which is 0-dimensional on A. Then there is $0 < \epsilon_{i+1} < 1/2^{i+1}$ such that (i) holds for $i+1$. Define $\omega_i^{i+1} = \tilde{\omega}_i^{i+1} \circ \pi_i : Y_{i+1} \longrightarrow Y_i$ and note that (ii) is satisfied. Clearly we may assume that Y_{i+1} admits a triangulation and the construction is completed.

Note that for every m-simplex Δ in Y_i we have that

$$H^m((\omega_i^{i+1})^{-1}(\Delta), (\omega_i^{i+1})^{-1}(\partial\Delta); \mathbb{Q}) =$$

$$H^m((\tilde{\omega}_i^{i+1})^{-1}(\Delta), (\tilde{\omega}_i^{i+1})^{-1}(\partial\Delta); \mathbb{Q}) = H^m(Y_\Delta, \partial\Delta; \mathbb{Q}) = 0.$$

This implies that for every subcomplex Z_i of Y_i we have that

$$H^m(Y_{i+1}, (\omega_i^{i+1})^{-1}(Z_i); \mathbb{Q}) = 0.$$

Recall that $\mathrm{diam}\,\omega_j^i(\Delta) < \epsilon_j/2^i < 1/2^{j+i}$ for every $i \geq j \geq 1$ and every simplex Δ of Y_i. Then $H^m(Y, Z; \mathbb{Q}) = 0$ for every closed subset Z of Y and hence $\dim_{\mathbb{Q}} Y \leq m-1$. The proposition is proved. ■

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